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On Some Relations for the Rank Moduli 9 and 12

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In this paper we prove some identities, conjectured by Lewis, concerning the rank moduli 9 and 12, which are similar to Dyson's identities for the rank moduli 5 and 7 which give a combinatorial explanation to Ramanujan's partition congruences. For this we use multisection of series and some identities for the third and sixth order Mock theta functions, in such a way that all the identities for a given modulus reduce to a single theta identity. © 1992 Academic Press, Inc.

INTRODUCTION

Let π denote a partition with the parts arranged in nonincreasing order. Dyson [4] defined the "rank" of a partition as the difference between the largest part and the number of parts. Denoting by $N(m, t, n)$ the number of partitions of n with rank congruent to m modulo t, he conjectured several relations between the numbers $N(m, t, kn+s)$ when $t=k=5$ and when $t=k=7$. Dyson conjectural identities are

$$N(1, 5, 5n+1) = N(2, 5, 5n+1) \tag{1}$$

$$N(0, 5, 5n+2) = N(2, 5, 5n+2) \tag{2}$$

$$N(0, 5, 5n+4) = N(1, 5, 5n+4) = \cdots = N(4, 5, 5n+4) \tag{3}$$

$$N(2, 7, 7n) = N(3, 7, 7n) \tag{4}$$

$$N(1, 7, 7n+1) = N(2, 7, 7n+1) = N(3, 7, 7n+1) \tag{5}$$

$$N(0, 7, 7n+2) = N(3, 7, 7n+2) \tag{6}$$

$$N(0, 7, 7n+3) = N(2, 7, 7n+3) \tag{7}$$

$$N(1, 7, 7n+3) = N(3, 7, 7n+3) \tag{8}$$

$$N(0, 7, 7n+4) = N(1, 7, 7n+4) = N(3, 7, 7n+4) \tag{9}$$

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$$N(0,7,7n+5)=N(1,7,7n+5)=\cdots=N(3,7,7n+5) \tag{10}$$

$$\begin{aligned} &N(0,7,7n+6)+N(1,7,7n+6)\\ &\quad=N(2,7,7n+6)+N(3,7,7n+6), \end{aligned} \tag{11}$$

all of which were proved by Atkin and Swinnerton-Dyer [3].

Of special importance are Eqs. (3) and (10) since they provide a combinatorial interpretation to Ramanujan congruences [8], viz. $p(5n+4)\equiv 0 \pmod 5$, and $p(7n+5)\equiv 0 \pmod 7$.

The purpose of this paper is to prove some identities stated by Lewis [7], similar to the ones found by Dyson, but related to the moduli 9 and 12. These identities are

$$N(3,9,3n)=N(4,9,3n) \tag{12}$$

$$\begin{aligned} &N(1,9,3n+1)+N(2,9,3n+1)\\ &\quad=N(3,9,3n+1)+N(4,9,3n+1) \end{aligned} \tag{13}$$

$$N(0,9,3n+2)=N(4,9,3n+2) \tag{14}$$

$$N(2,12,2n)=N(5,12,2n) \tag{15}$$

$$N(1,12,2n+1)=N(4,12,2n+1) \tag{16}$$

To prove these identities we first apply partial fractions decomposition and multisection of series in such a way that all the identities related to a given modulus are proved to be equivalent to a single identity. Using different results from the literature each of these two identities can be shown to be equivalent to a theta identity, that then is proved.

Preliminary Results

In this section we present most of the results that will be needed to prove Eqs. (12) to (16). The following theorem appears in Riordan's book [9, p. 131],

THEOREM 1. *If $f(q)$ is the generating function for the coefficients A_n, this is*

$$f(q)=\sum_{n=0}^{\infty} A_n q^n$$

and ξ is a primitive t-root of 1, then

$$\sum_{m=0}^{\infty} A_{tm+r}q^{tm+r}=\frac{1}{t}\sum_{k=0}^{t-1}\xi^{(t-r)k}f(\xi^k q).$$

From [3] we need the following results. Define

$$S_b(\alpha)=\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2+\alpha n}}{1-q^{bn}}. \tag{17}$$

Writing $-n$ for n in (17) we have

$$S_b(\alpha)=-S_b(b-1-\alpha). \tag{18}$$

Also we need the generating function for $N(m, t, n)$, given by

$$\sum_{n=0}^{\infty} N(m, t, n)q^n=\frac{1}{(q;q)_\infty}\sum_{n\neq 0}\frac{(-1)^n q^{n(3n+1)/2}[q^{mn}+q^{(t-m)n}]}{1-q^{tn}}. \tag{19}$$

From [10] we need the following definitions and results, concerning the third order Mock theta functions:

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}=\frac{2}{(q;q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n} \tag{20}$$

$$\phi_3(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_n}=\frac{2}{(q;q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}} \tag{21}$$

$$\begin{aligned}\chi(q)&=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-q+q^2)\cdots(1-q^n+q^{2n})}\\&=\frac{1}{(q;q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}}{1-q^n+q^{2n}}\end{aligned} \tag{22}$$

$$\begin{aligned}1+\sum_{n=1}^{\infty}&\frac{(-1)^n(1+q^n)(2-2\cos\theta)q^{n(3n+1)/2}}{1-2q^n\cos\theta+q^{2n}}\\&=(q)_\infty\left[1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(e^{i\theta}q)_n(e^{-i\theta}q)_n}\right].\end{aligned} \tag{23}$$

Note that (23) is used to prove the last equalities of (20) to (22).

Among the relations proved by Watson [10] we need the following ones

$$2\phi_3(-q)-f(q)=\frac{(q;q)_\infty}{(-q;q)_\infty^2} \tag{24}$$

$$4\chi(q)-f(q)=\frac{3(q^3;q^3)_\infty^2}{(q;q)_\infty(-q^3;q^3)_\infty^2}. \tag{25}$$

Also we need some of the sixth order Mock theta functions and its relations, as exhibited by Andrews and Hickerson [2]. Define

$$\phi_6(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q; q^2)_n}{(-q; q)_{2n}} \tag{26}$$

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q; q)_n}{(q^3; q^3)_n}. \tag{27}$$

In [2] they proved

$$2\gamma(q) = 3\phi_6(q) - \frac{j(q, q^2)^2}{j(-q, q^3)}, \tag{28}$$

where $j(x, q)$ is defined (for $x \neq 0$ and $|q| < 1$) as

$$j(x, q) = (x; q)_\infty (q/x; q)_\infty (q; q)_\infty = \sum_n (-1)^n q^{n(n-1)/2} x^n. \tag{29}$$

We use the following notation, first introduced by Hickerson [6]:

$$J_{a,m} = j(q^a, q^m)$$
$$\bar{J}_{a,m} = j(-q^a, q^m)$$
$$J_m = J_{m,3m} = (q^m; q^m)_\infty.$$

Next, we define $R(z)$ as

$$R(z) = \sum_n \frac{(-1)^n q^{n(3n+1)/2} z^n}{1 - zq^{3n}}. \tag{30}$$

Then, from (3.15) and (3.18) of [2] we obtain

$$\frac{2R(z)}{j(zq^2, q^3)} = \phi_6(q) + \frac{J_1 J_{3,6}\, j(-z, q^3)\, j(-zq^2, q^3)}{J_2\, j(z, q^3)\, j(zq^2, q^3)} \tag{31}$$

Finally we quote Theorems (1.1) and (1.2) of [2]:

THEOREM 2. *If n is a positive integer, $0 < |q| < 1$, and $x \neq 0$, then*

$$j(x, q) = \sum_{k=0}^{n-1} (-1)^k q^{k(k-1)/2} x^k j[(-1)^{n+1} q^{n(n-1)/2 + kn} x^n, q^{n^2}]. \tag{32}$$

In particular if $n = 2$ we obtain

$$j(x, q) = j(-x^2 q, q^4) - x j(-x^2 q^3, q^4). \tag{33}$$

THEOREM 3. *If n is a positive integer, $0<|q|<1$, and neither x nor z is 0 or an integral power of q, then*

$$\frac{J_1^3\, j(xz, q)\, j(x^n, q^n)}{J_n^3\, j(x, q)\, j(z, q)} = \sum_{k=0}^{n-1} \frac{x^k j(q^k x^n z, q^n)}{j(q^k z, q)}. \tag{34}$$

COROLLARY 3.1.

$$\frac{\omega j(-\omega^2 q, q^3)}{j(\omega^2 q, q^3)} + \frac{\omega^3 j(-\omega q, q^3)}{j(\omega q, q^3)} = -\frac{J_9^3 \bar{J}_{0,3} J_3^2 J_{2,3}^2}{J_3^4 \bar{J}_{0,9} \bar{J}_{3,9} \bar{J}_{1,3}^2}. \tag{35}$$

Proof. In Theorem 3 replace q by q^3, and let $x=\omega q$, $z=-1$, and $n=3$ (where ω is a primitive cubic root of 1), then we have

$$\frac{J_3^3\, j(-\omega q, q^3) J_{3,9}}{J_9^3\, j(\omega q, q^3) \bar{J}_{0,3}} = \frac{\bar{J}_{3,9}}{\bar{J}_{0,9}} + \omega q \frac{\bar{J}_{6,9}}{\bar{J}_{3,9}} + \omega^2 q^2 \frac{\bar{J}_{9,9}}{\bar{J}_{6,9}}. \tag{36}$$

Since

$$J_{3,9} = J_3, \qquad \bar{J}_{0,9} = \bar{J}_{9,9}, \qquad \bar{J}_{3,9} = \bar{J}_{6,9}$$

we can rewrite (36) as

$$\frac{\omega^2 j(-\omega q, q^3)}{j(\omega q, q^3)} = \frac{J_9^3 \bar{J}_{0,3}(\omega^2 \bar{J}_{6,9}^2 + q\bar{J}_{6,9}\bar{J}_{9,9} + \omega q^2 \bar{J}_{9,9}^2)}{J_3^4 \bar{J}_{0,9} \bar{J}_{3,9}}. \tag{37}$$

If in (37) we replace ω by ω^2 and add the resulting equation to (37), we obtain

$$\frac{\omega j(-\omega^2 q, q^3)}{j(\omega^2 q, q^3)} + \frac{\omega^2 j(-\omega q, q^3)}{j(\omega q, q^3)} = -\frac{J_9^3 \bar{J}_{0,3}}{J_3^4 \bar{J}_{0,9} \bar{J}_{3,9} \bar{J}_{1,3}^2} [\bar{J}_{6,9} - q\bar{J}_{9,9}]^2.$$

But, if in Theorem 1.0 of [6] (the quintuple product identity expressed in "j" notation) q is replaced by q^3 and we let $x=-q$, we obtain

$$\bar{J}_{6,9} - q\bar{J}_{9,9} = \frac{J_3 J_{2,3}}{\bar{J}_{1,3}}. \tag{39}$$

Replacing (39) in (38) we obtain the desired result.

THE LEWIS CONJECTURES MODULUS 9

To prove Eqs. (12) to (14) we first study the generating functions for the corresponding differences. By (19) the generating function for $N(3, 9, n) - N(4, 9, n)$ equals

$$N_0(q) = \frac{1}{(q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(3n+1)/2}(q^{3n} + q^{6n} - q^{4n} - q^{5n})}{1-q^{9n}} \tag{40}$$

but, from (18) we know that $S_9(4)=0$, and $S_9(3)=-S_9(5)$. Moreover, as

$$\frac{q^{3n}+q^{6n}-q^{4n}-q^{5n}}{1-q^{9n}}=\frac{q^{3n}(1-q^{2n})}{(1+q^n+q^{2n})(1+q^{3n}+q^{6n})}$$

the term which corresponds to $n=0$ can be included in the summation, since this term is equal to 0. From all these (40) can be rewritten as

$$N_0(q)=\frac{1}{(q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}(2q^{3n}+q^{6n})}{1-q^{9n}}, \tag{41}$$

but

$$\frac{2q^{3n}+q^{6n}}{1-q^{9n}}=\frac{1}{1-q^{3n}}-\frac{1}{1+q^{3n}+q^{6n}}.$$

Thus, if we define

$$T(q)=\frac{1}{(q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}}{1-q^{3n}} \tag{42}$$

$$K_\alpha(q)=\frac{1}{(q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}q^{\alpha n}}{1+q^{3n}+q^{6n}} \tag{43}$$

we can express $N_0(q)$ as

$$N_0(q)=T(q)-K_0(q). \tag{44}$$

In a similar fashion we obtain that the generating function for $N(3,9,n)+N(4,9,n)-N(1,9,n)-N(2,9,n)$, denoted by $N_1(q)$, is equal to

$$N_1(q)=T(q)-K_3(q). \tag{45}$$

Finally, the generating function for $N(0,9,n)-N(4,9,n)$, denoted by $N_2(q)$, is equal to

$$N_2(q)=T(q)-K_6(q). \tag{46}$$

Before continuing we note that

$$K_0(q)+K_3(q)+K_6(q)=\frac{1}{(q)_\infty}\sum_n(-1)^n q^{n(3n+1)/2}=1. \tag{47}$$

Now, if ω is a primitive cubic root of 1 and we apply Theorem 1 to the generating functions of the differences corresponding to Eqs. (12) to (14), we find that (12) to (14) are equivalent, respectively, to

$$N_0(q)+N_0(\omega q)+N_0(\omega^2 q)=0 \tag{48}$$

$$N_1(q)+\omega^2 N_1(\omega q)+\omega N_1(\omega^2 q)=0 \tag{49}$$

$$N_2(q)+\omega N_2(\omega q)+\omega^2 N_2(\omega^2 q)=0. \tag{50}$$

Replacing (44) to (46) in (48) to (50), we obtain

$$T(q)+T(\omega q)+T(\omega^2 q)=K_0(q)+K_0(\omega q)+K_0(\omega^2 q) \tag{51}$$

$$T(q)+\omega^2 T(\omega q)+\omega T(\omega^2 q)=K_3(q)+\omega^2 K_3(\omega q)+\omega K_3(\omega^2 q) \tag{52}$$

$$T(q)+\omega T(\omega q)+\omega^2 T(\omega^2 q)=K_6(q)+\omega K_6(\omega q)+\omega^2 K_6(\omega^2 q). \tag{53}$$

Equations (51) to (53) can be considered as a system of 3 equations with 3 unknowns. Solving it and noting that $T(wq)$ and $T(w^2q)$ depend on $T(q)$, we obtain that (51) to (53) are equivalent, all together, to

$$\begin{aligned}3T(q)=1&+K_0(\omega q)+\omega^2 K_3(\omega q)+\omega K_6(\omega q)+K_0(\omega^2 q)\\&+\omega K_3(\omega^2 q)+\omega^2 K_6(\omega^2 q).\end{aligned} \tag{54}$$

From (47) we find

$$K_6(\omega q)=1-K_0(\omega q)-K_3(\omega q) \tag{55}$$

$$K_6(\omega^2 q)=1-K_0(\omega^2 q)-K_3(\omega^2 q). \tag{56}$$

Replacing (55) and (56) in (54) we obtain, after some simplification

$$T(q)=\frac{1-\omega}{3}\left[(K_0(\omega q)-\omega K_3(\omega q))+\omega(K_3(\omega^2 q)-\omega K_0(\omega^2 q))\right]. \tag{57}$$

On the other hand, from (43) we know that

$$\begin{aligned}K_0(q)&=\frac{1}{(q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{3n}+q^{6n}}\\&=\frac{1}{(\omega-\omega^2)(q)_\infty}\sum_n(-1)^n q^{n(3n+1)/2}\left[\frac{\omega}{1-\omega q^{3n}}-\frac{\omega^2}{1-\omega^2 q^{3n}}\right]\end{aligned} \tag{58}$$

$$K_3(q)=\frac{1}{(\omega-\omega^2)(q)_\infty}\sum_n(-1)^n q^{n(3n+1)/2}\left[\frac{1}{1-\omega q^{3n}}-\frac{1}{1-\omega^2 q^{3n}}\right]. \tag{59}$$

From (58) and (59) we obtain

$$K_0(q)-\omega K_3(q)=\frac{1}{(q;q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}}{1-\omega^2 q^{3n}} \tag{60}$$

$$\omega[K_3(q)-\omega K_0(q)]=\frac{-\omega^2}{(q;q)_\infty}\sum_n\frac{(-1)^n q^{n(3n+1)/2}}{1-\omega q^{3n}}. \tag{61}$$

Using (60), (61), and the fact that $\omega^{n(3n+1)/2}=\omega^{2n}$, (54) can be rewritten as

$$\frac{6T(q)}{1-\omega}=\frac{2R(\omega^2)}{(\omega q;\omega q)_\infty}-\frac{2\omega^2 R(\omega)}{(\omega^2 q;\omega^2 q)_\infty} \tag{62}$$

where $R(z)$ was defined in (30).

If in (31) we let $z=\omega$ and $z=\omega^2$, we obtain, respectively, the equations

$$\frac{2R(\omega)}{j(\omega q^2;q^3)}=\phi_6(q)+\frac{J_1J_{3,6}\,j(-\omega,q^3)\,j(-\omega q^2,q^3)}{J_2\,j(\omega,q^3)\,j(\omega q^2,q^3)} \tag{63}$$

$$\frac{2R(\omega^2)}{j(\omega^2q^2;q^3)}=\phi_6(q)+\frac{J_1J_{3,6}\,j(-\omega^2,q^3)\,j(-\omega^2q^2,q^3)}{J_2\,j(\omega^2,q^3)\,j(\omega^2q^2,q^3)}. \tag{64}$$

From the definition of $j(x,q)$ we also obtain

$$\begin{aligned} j(\omega q^2,q^3)&=j(q^3/\omega q^2,q^3)=j(\omega^2 q,q^3)\\ &=(\omega^2 q,(\omega^2 q)^3)_\infty\,((\omega^2q)^2,(\omega^2q)^3)_\infty\,((\omega^2q)^3,(\omega^2q)^3)_\infty\\ &=(\omega^2 q,\omega^2 q)_\infty \end{aligned} \tag{65}$$

and similarly

$$j(\omega^2q^2,q^3)=j(q^3/\omega^2q^2,q^3)=j(\omega q,q^3)=(\omega^2 q,\omega^2 q)_\infty. \tag{66}$$

Replacing (63) to (66) in (62) we obtain

$$\begin{aligned} 6T(q)&=\phi_6(q)(1-\omega^2)(1-\omega)\\ &\quad+\frac{(1-\omega)J_1J_{3,6}}{J_2}\left[\frac{j(-\omega^2,q^3)\,j(-(\omega^2q^2,q^3)}{j(\omega^2,q^3)\,j(\omega^2q^2,q^3)}\right.\\ &\quad\left.-\frac{\omega^2 j(-\omega,q^3)\,j(-\omega q^2,q^3)}{j(\omega,q^3)\,j(\omega q^2,q^3)}\right]. \end{aligned} \tag{67}$$

But

$$(1-\omega)(1-\omega^2)=1-(\omega+\omega^2)+\omega^3=3$$

and

$$j(x, q^3) = j(q^3/x, q^3).$$

Also

$$\frac{(1-\omega)\, j(-\omega^2, q^3)}{j(\omega^2, q^3)}$$

$$= \frac{(1-\omega)(1+\omega^2)(-\omega^2 q^3; q^3)_\infty\, (-\omega q^3; q^3)_\infty\, J_3(-q^3; q^3)_\infty}{(1-\omega^2)(\omega^2 q^3; q^3)_\infty\, (\omega q^3; q^3)_\infty\, J_3(-q^3; q^3)_\infty}$$

$$= \frac{\omega^2(-q^9; q^9)_\infty J_3}{J_9(-q^3; q^3)_\infty} \tag{68}$$

and

$$\frac{-\omega^2(1-\omega)\, j(-\omega, q^3)}{j(\omega, q^3)} = \frac{\omega(-q^9; q^9)_\infty J_3}{J_9(-q^3; q^3)_\infty}. \tag{69}$$

Using all these results we can rewrite (67) as

$$6T(q) = 3\phi_6(q) + \frac{J_1 J_{3,6} J_{18} J_3^1}{J_2 J_9^2 J_6} \left[\frac{\omega j(-\omega^2 q, q^3)}{j(\omega^2 q, q^3)} + \frac{\omega^2 j(-\omega q, q^3)}{j(\omega q, q^3)} \right]. \tag{70}$$

But, by Corollary (3.1) the term inside the brackets equals

$$-\frac{J_9^3 \bar{J}_{0,3} J_3^2 J_{2,3}^2}{J_3^4 \bar{J}_{0,9} \bar{J}_{3,9} \bar{J}_{1,3}^2}.$$

Therefore the second term on the right-hand side of (70) becomes

$$-\frac{J_1 J_{3,6} J_{18} J_3^2 J_9^3 \bar{J}_3^2 J_{2,3}^2}{J_2 J_9^2 J_6 J_3^4 \bar{J}_{0,9} \bar{J}_{3,9} \bar{J}_{1,3}^2} = -\frac{J_1^5 J_6}{J_2^3 J_3^3} = -\frac{J_1^4/J_2^2}{J_3^2 J_2/J_1 J_6} = -\frac{j^2(q, q^2)}{j(-q, q^3)}. \tag{71}$$

Replacing (71) in (70) we obtain

$$6T(q) = 3\phi_6(q) - \frac{j^2(q, q^2)}{j(-q, q^3)}. \tag{72}$$

Comparing (72) with (28) we realize that to prove (12) to (14) it only remains to prove

$$6T(q) = 2\gamma(q). \tag{73}$$

Now, if in (23) we let $\theta = 2\pi/3$, then we obtain

$$1 + 3\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^n+q^{2n}} = (q; q)_\infty \left[1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\omega q)_n (\omega^{-1} q)_n} \right]. \tag{74}$$

Writing $-n$ for n we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2} q^n}{1+q^n+q^{2n}} = \sum_{n=-\infty}^{-1} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n+q^{2n}}$$

and if we note that the term which corresponds to $n=0$ has a value of $\frac{1}{3}$, then we can write the left-hand side of (74) as

$$1+3\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^n+q^{2n}} = 3\sum_{n} \frac{(-1)^n q^{n(3n+1)/2}(1-q^n)}{(1+q^n+q^{2n})(1-q^n)}$$

$$3\sum_{n} \frac{(-1)^n q^{n(3n+1)/2}}{1-q^{3n}} - 3\sum_{n} \frac{(-1)^n q^{3n(n+1)/2}}{1-q^{3n}} = 3T(q)(q;q)_\infty, \tag{75}$$

where the last line follows from (18), since the second term is equal to $S_3(1)$.

On the other hand

$$1+\sum_{n=1}^{\infty} \frac{q^{n^2}}{(\omega q)_n(\omega^{-1}q)_n} = \sum_{n=0}^{\infty} \frac{q^{n^2}(q)_n}{(\omega q)_n(\omega^{-1}q)_n(q)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2}(q)_n}{(q^3;q^3)_n} = \gamma(q). \tag{76}$$

Replacing (75) and (76) in (73) we obtain (72) and in this way we conclude that Eqs. (12) to (14) are true.

The Lewis Conjectures Modulus 12

To prove Eqs. (15) and (16) we first study the generating functions for the corresponding differences. Now the generating functions for $N(1,12,n)-N(4,12,n)$ is given by

$$g(q) = \frac{1}{(q)_\infty} \sum_{n\neq 0} \frac{(-1)^n q^{n(3n+1)/2}(q^n+q^{11n}-q^{4n}-q^{8n})}{1-q^{12n}}$$

$$= -\frac{1}{(q)_\infty} \sum_{n\neq 0} \frac{(-1)^n q^{n(3n+1)/2}(q^{10n}+1-q^{7n}-q^{3n})}{1-q^{12n}}, \tag{77}$$

where the last line is obtained replacing $-n$ for n. But,

$$\frac{q^{10n}+1-q^{7n}-q^{3n}}{1-q^{12n}} = \frac{(1-q^{3n})(1-q^{7n})}{(1-q^{3n})(1+q^{3n})(1+q^{6n})} = \frac{1-q^{7n}}{(1+q^{3n})(1+q^{6n})}$$

and thus (77) can be rewritten as

$$g(q) = -\frac{1}{(q)_\infty}\sum_n \frac{(-1)^n q^{n(3n+1)/2}(1-q^{7n})}{(1+q^{3n})(1+q^{6n})}, \tag{78}$$

where we include the term for $n=0$ since this is equal to 0.

Now, we note that if we write $-n$ for n

$$\sum_n \frac{(-1)^n q^{n(3n+1)/2} q^{7n}}{(1+q^{3n})(1+q^{6n})} = \sum_n \frac{(-1)^n q^{n(3n+1)/2} q^n}{(1+q^{3n})(1+q^{6n})}. \tag{79}$$

Replacing (79) in (78) and using the fact that

$$\frac{1}{(1+q^{3n})(1+q^{6n})} = \frac{1}{2}\left[\frac{1}{1+q^{3n}} + \frac{1-q^{3n}}{1+q^{6n}}\right];$$

we can rewrite (78) as

$$\begin{aligned} g(q) &= -\frac{1}{2(q)_\infty}\sum_n (-1)^n q^{n(3n+1)/2}(1-q^n)\left[\frac{1}{1+q^{3n}} + \frac{1-q^{3n}}{1+q^{6n}}\right] \\ &= -\frac{1}{2(q)_\infty}\left[\sum_n \frac{(-1)^n q^{n(3n+1)/2}(1-q^n)}{1+q^{3n}}\right. \\ &\quad \left. + \sum (-1)^n q^{n(3n+1)/2}\left[\frac{1-q^n-q^{3n}+q^{4n}}{1+q^{6n}}\right]\right] \end{aligned} \tag{80}$$

but writing $-n$ for n we have

$$\sum_n \frac{(-1)^n q^{n(3n+1)/2} q^n}{1+q^{6n}} = \sum_n \frac{(-1)^n q^{n(3n+1)/2} q^{4n}}{1+q^{6n}}. \tag{81}$$

Therefore $g(q)$ can be expressed as

$$g(q) = -\tfrac{1}{2}[A(q) + B(q)], \tag{82}$$

where

$$A(q) = \frac{1}{(q;q)_\infty}\sum_n \frac{(-1)^n q^{n(3n+1)/2}(1-q^n)}{1+q^{3n}}$$

and

$$B(q) = \frac{1}{(q;q)_\infty}\sum_n \frac{(-1)^n q^{n(3n+1)/2}(1-q^{3n})}{1+q^{6n}}.$$

Similarly, the generating function for the difference $N(2, 12, n)-N(5, 12, n)$, denoted by $h(q)$, can be written as

$$h(q)=-\tfrac{1}{2}[-A(q)+B(q)]. \tag{83}$$

In terms of the generating function (15) is equivalent to saying that $g(q)$ is an even function, which means

$$g(q)-g(-q)=0. \tag{84}$$

Also (16) is equivalent to saying that $h(q)$ is an odd function, which means

$$h(q)+h(-q)=0. \tag{85}$$

Adding (84) and (85) side by side and simplifying similar terms, we obtain (after exchanging q by $-q$)

$$A(q)=B(-q). \tag{86}$$

Replacing (86) in (83) and (81) we realize that in order to prove (15) and (16) it is sufficient to demonstrate the truth of (86).

Next, we write $A(q)$ and $B(q)$ in terms of some of the third order Mock theta functions. For this note that

$$\frac{1-q^n}{1+q^{3n}}=\frac{1}{3}\left[\frac{2}{1+q^n}+\frac{1-2q^n}{1-q^n+q^{2n}}\right] \tag{87}$$

and writing $-n$ for n we obtain

$$\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1-q^n+q^{2n}}=\sum_n \frac{(-1)^n q^{n(3n+1)/2} q^n}{1-q^n+q^{2n}}. \tag{88}$$

Replacing (87) and (88) in the definition of $A(q)$ we obtain

$$\begin{aligned} A(q)&=\frac{1}{3(q;q)_\infty}\left[2\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}-\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1-q^n+q^{2n}}\right] \\ &=\frac{1}{3}[f(q)-\chi(q)] \end{aligned} \tag{89}$$

Similarly, using

$$\sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{6n}}=\sum_n \frac{(-1)^n q^{n(3n+1)/2} q^{5n}}{1+q^{6n}} \tag{90}$$

we can rewrite $B(q)$ as

$$B(q) = \frac{1}{(q;q)_\infty} \sum \frac{(-1)^n q^{n(3n+1)/2}(q^{5n} - q^{3n})}{1+q^{6n}}$$
$$= \frac{1}{(q;q)_\infty} \left[\sum_n \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{2n}} - \sum_n \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{6n}} \right]. \tag{91}$$

But Eq. (7.15) of Garvan [5] establishes

$$1 + (1-z)(1-z^{-1}) \sum_{n=1}^{\infty} \frac{(-1)^n q^{(n+1)/2}(1+q^n)}{1-(z+z^{-1})q^n + q^{2n}} = \frac{(q;q)_\infty^2}{(zq;q)_\infty (z^{-1}q;q)_\infty}.$$

Letting $z = i$ in (92) and rewriting the left-hand side of the resulting equation as a bilateral series we obtain

$$2 \sum_n \frac{(-1)^n q^{n(n+1)/2}}{1+q^{2n}} = \frac{(q;q)_\infty^2}{(iq;q)_\infty (iq;q)_\infty}$$
$$= \frac{(q;q)_\infty^2}{(-q^2;q^2)_\infty} = J_1^2(q^2;q^4)_\infty. \tag{93}$$

On the other hand if we write $-n$ for n, we have

$$\frac{1}{(q;q)_\infty} \sum_n \frac{(-1)^n q^{3n(n+1)/2}}{1+q^{2n}}$$
$$= \frac{1}{(q;q)_\infty} \sum_n \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{2n}} = \frac{1}{2} \phi_3(q). \tag{94}$$

Using (93), with q replaced by q^3, and (94) we can express $B(q)$ as

$$B(q) = \tfrac{1}{2}[\phi_3(q) - \Pi(q)], \tag{95}$$

where

$$\Pi(q) = \frac{(q^3;q^3)_\infty^2}{(q;q)_\infty (-q^6;q^6)_\infty}.$$

Using (89) and (95) we can rewrite (85) as

$$4\chi(q) - f(q) + 3[2\phi_3(-q) - f(q)] = 6\Pi(-q). \tag{96}$$

Using (24) and (25) we obtain, after multiplying by $(q;q)_\infty$, that (96) is equivalent to

$$\frac{(q^3;q^3)_\infty^2}{(-q^3;q^3)_\infty^2} + \frac{(q;q)_\infty^2}{(-q;q)_\infty^2} = 2\Pi(-q)(q;q)_\infty. \tag{97}$$

But,

$$2\Pi(-q)(q;q)_\infty = 2\frac{(q;q)_\infty(-q^3;-q^3)_\infty^2}{(-q;-q)_\infty(-q^6;q^6)_\infty}$$

$$= 2\frac{(q;q^2)_\infty(-q^3;q^6)_\infty^2(q^6;q^6)_\infty^2}{(-q;q^2)_\infty(-q^6;q^6)_\infty} \tag{98}$$

and

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = \frac{(q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty}{(-q;q^3)_\infty(-q^2;q^3)_\infty(-q^3;q^3)_\infty}. \tag{99}$$

Using (98) and (99), and multiplying (97) by the inverse of the first term on the right-hand side, we obtain

$$1+\left[\frac{(q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty}{(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty}\right]^2$$

which can be rewritten in "j" notation as

$$2\left[\frac{(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty}{(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty}\right]\frac{(q^6;q^6)_\infty^2(-q^3;q^3)_\infty^2}{(-q^6;q^6)_\infty^2(q^3;q^3)_\infty^2} \tag{100}$$

$$1+\left[\frac{J_{1,3}}{\bar{J}_{1,3}}\right]^2 = 2\frac{J_{1,6}J_6^6}{\bar{J}_{1,6}J_{12}^2J_3^4} \tag{101}$$

or equivalently

$$\frac{[[J_{1,3}]^2+[\bar{J}_{1,3}]^2]\bar{J}_{1,6}}{2J_{1,6}\bar{J}_{1,3}^2} = \frac{J_6^6}{J_{12}^2J_3^4}. \tag{102}$$

By Theorem 1.1 of [6] letting $y=x$ we have

$$j(-x,q)^2 = j(-x^2,q^2)\,j(-q,q^2) + xj(-x^2q,q^2)\,j(-1,q^2). \tag{103}$$

Replacing x by $-x$ in (103) and adding the resulting equation to (103) we get

$$j(-x,q)^2 + j(x,q)^2 = 2j(-x^2,q^2)\,j(-q,q^2). \tag{104}$$

Replacing q^3 for q and q for x in (104), we obtain

$$J_{1,3}^2 + \bar{J}_{1,3}^2 = 2\bar{J}_{2,6}\bar{J}_{3,6}. \tag{105}$$

Replacing (105) in (102) we obtain

$$\frac{\bar{J}_{1,6}\bar{J}_{2,6}\bar{J}_{3,6}}{J_{1,6}\bar{J}_{1,3}^2} = \frac{J_6^6}{J_{12}^2J_3^4}. \tag{106}$$

But, the left-hand side of (106) is equal to

$$L = \frac{(-q;q^6)_\infty(-q^5;q^6)_\infty(-q^2;q^6)_\infty(-q^4;q^6)_\infty(-q^3;q^6)_\infty^2 J_6^3}{(q;q^6)_\infty(q^5;q^6)_\infty(-q;q^3)_\infty^2(-q^2;q^3)_\infty^2 J_3^2 J_6} \tag{107}$$

and

$$(-q;q^3)_\infty(-q^2;q^3)_\infty = (-q;q^6)_\infty(-q^4;q^6)_\infty(-q^2;q^6)_\infty(-q^5;q^6)_\infty. \tag{108}$$

Simplifying similar terms and multiplying numerator and denominator by $(q^3;q^6)_\infty(-q^3;q^3)_\infty$ we obtain

$$L = \frac{J_6^2(-q;q^6)_\infty(q^6;q^{12})_\infty(-q^3;q^3)_\infty}{J_3^2(q;q^2)_\infty(-q;q)_\infty} \tag{109}$$

Using Corollary (1.2) of [1], (109) becomes

$$L = \frac{J_6^2(-q^3;q^3)_\infty(-q^3;q^6)_\infty(-q^6;q^6)_\infty}{J_3^2(-q^6;q^6)_\infty^2} = \frac{J_6^2(-q^3;q^3)_\infty^2}{J_3^2(-q^6;q^6)_\infty^2}$$

$$= \left[\frac{J_6^2}{J_{12}J_3}\right]^2 \tag{110}$$

where in the last line we use the fact that

$$(-q^m;q^m) = \frac{J_{2m}}{J_m}.$$

Therefore the left- and right-hand sides of (106) are equal. Hence we conclude that (15) and (16) are true.

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